W13/2 Dedekind domains, Class group

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Prop 8.9 A Dedekind domain, $K = \mathcal{F}(A)$. Then every invertible frac. $I \neq 0$ has a unique repr (up to permutation) $I = P_1^{k_1} \cdots P_r^{k_r}$, $P_i \in \operatorname{Max}(A)$ pw. distinct, $k_i \in \mathbb{Z}$.

Proof: Let $0 \neq a \in A$ s.t. $\underbrace{aI}_{\neq 0} \subseteq A$. Factor $aI = P_1^{m_1} \cdots P_r^{m_r}$ $(m_i \geq 0, r \geq 0)$

and $(a) = P_1^{n_1} \cdots P_r^{n_r}$ $(n_i \geq 0)$ w. pw. distinct $P_1, \dots, P_r \in \operatorname{Max}(A)$ (P8.5).

$\Rightarrow I = (a)^{-1} I = P_1^{-n_1} \cdots P_r^{-n_r} P_1^{m_1} \cdots P_r^{m_r} = P_1^{m_1 - n_1} \cdots P_r^{m_r - n_r}$

Uniqueness: If $P_1^{k_1} \cdots P_r^{k_r} = P_1^{e_1} \cdots P_r^{e_r}$ with $k_i, e_i \in \mathbb{Z}$,

cross-multiply the negative powers & apply P8.5(2). $\square$

Def: For A a Dedekind domain, $P \in \operatorname{Spec}(A) \setminus \{0\}$, I invertible frac. ideal, let $v_P(I) \in \mathbb{Z}$ be the exponent of P in the prime fact. of I.

Remark: (1) A_P is a dvr, so P gives rise to a d.v. $v_P : K \to \mathbb{Z} \cup \{\infty\}$, (same notation)

For $a \in K^\times$, then $v_P(a) = v_P(\underbrace{(a)}_{\text{principal ideal}})$.

(2) $v_P(IJ) = v_P(I) + v_P(J)$

Cor 8.10: Let A be a Dedekind domain, G the group of invertible frac. ideals. Then

$$\begin{cases} (G, \cdot) \xrightarrow{\sim} (\mathbb{Z}^{(\operatorname{Spec}(A) \setminus \{0\})}, +) = \bigoplus_{P \in \operatorname{Spec}(A) \setminus \{0\}} \mathbb{Z} \\ I \longmapsto (v_P(I))_{P \in \operatorname{Spec}(A) \setminus \{0\}} \end{cases}$$

is a group isomorphism.

Proof: Group hom since $v_P(IJ) = v_P(I) + v_P(J)$.

Bijective by P.8.9. $\square$

For $0 \neq I$ fractional: $I = \prod_{P \in \operatorname{Spec}(A) \setminus \{0\}} P^{v_P(I)}$.

Lemma 8.11 A Dedekind domain, $P \in \operatorname{Spec}(A) \setminus \{0\}$, I, J invertible frac. ideals.

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$\Rightarrow v_P(I+J) = \min\{v_P(I), v_P(J)\}$, $v_P(I \cap J) = \max\{v_P(I), v_P(J)\}$

Proof: Can work on the DVR A_P, with uniformizer $\pi \in A_P$

$IA_P = P^e A_P = \pi^e A_P$ with $e = v_P(I)$,

$JA_P = \pi^f A_P$ with $f = v_P(J)$

Fractional ideals of A_P: $\{\pi^k A_P : k \in \mathbb{Z}\}$, $\pi^k A_P \subseteq \pi^\ell A_P \Leftrightarrow k \geq \ell$.

$\rightarrow \pi^e A_P + \pi^f A_P = \pi^{\min\{e,f\}} A_P$, $\pi^e A_P \cap \pi^f A_P = \pi^{\max\{e,f\}} A_P$. $\square$

Def: A Dedekind domain, G group of nonzero frac. ideals, H subgroup of $\neq 0$ principal frac. ideals.

Then $\mathcal{C}(A) := G/H$ is the (ideal) class group = divisor class group = Picard group (notation: $\operatorname{Pic}(A)$) of A.

Thm 8.12 A Dedekind domain. TFAE:

(a) A is a PID

(b) A is a UFD

(c) $\mathcal{C}(A)$ is trivial

Proof: (a) $\Rightarrow$ (b) ✓ (b) $\Rightarrow$ (c) Let G be the group of nonzero frac. ideals, $G \to \mathcal{C}(A)$, $I \mapsto [I]$ canonical epi. Since G is gen. by prime ideals (P 8.9), $\mathcal{C}(A)$ is generated by $\{[P] : P \in \operatorname{Spec}(A) \setminus \{0\}\}$

Suffices: every $P \in \operatorname{Spec}(A) \setminus \{0\}$ is principal.

Let $0 \neq a \in P \overset{(b)}{\Rightarrow} a = p_1 \cdots p_k$ with $k \geq 1$, prime elts. p_i

$\xRightarrow{P \text{ prime}} \exists i : p_i \in P$, soy $p_1 \in P$.

$\Rightarrow 0 \subsetneq (p_1) \subseteq P$ is a chain of prime ideals $\xRightarrow{\dim A = 1} P = (p_1)$.

(c) $\Rightarrow$ (a) If $0 \neq I \subsetneq A$, then $[I] \in \mathcal{C}(A)$ is trivial, i.e., $I \in H$, with H the subgroup of principal frac. ideals $\rightarrow I = Ax$, $0 \neq x \in K$.

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$I \subseteq A \Rightarrow x \in A$. $\square$

⚠ $K[x_1, \dots, x_n]$, K field, is a UFD, but a PID/Dedekind domain only if $n \leq 1$.

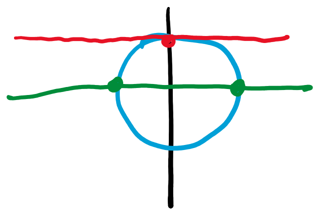
Exm (1) K alg. closed, $X \subseteq \mathbb{A}^n$ irred. curve, $A = A(X)$

$\{\text{points on } X\} \leftrightarrow \operatorname{Max}(A)$, $\underline{a} = (a_1, \dots, a_n) \mapsto (\bar{x}_1 - a_1, \dots, \bar{x}_n - a_n) =: I(\underline{a})$

$f \in A$ is a function $f: X \to K$: $f(\underline{a}) = 0 \Leftrightarrow f \in I(\underline{a})$

So: $$(f) = \prod_{\substack{\underline{a} \in X \\ f(\underline{a}) = 0}} I(\underline{a})^{v_{I(a)}(f)}$$ where $v_{I(a)}(f)$ ... order of vanishing of f at O.

E.g.: $\mathbb{C}[x,y]/(x^2+y^2-1) = A$



i) $f = \bar{y}$ $\qquad (\bar{y}) = (\bar{x}-1, \bar{y})(\bar{x}+1, \bar{y})$

["$\supseteq$": Only $(\bar{x}-1)(\bar{x}+1) \in (\bar{y})$ is non-trivial, and

$(\bar{x}-1)(\bar{x}+1) = \bar{x}^2 - 1 = -\bar{y}^2 \in (y)$

"$\subseteq$": $\frac{1}{2}[-(\bar{x}-1)\bar{y} + (\bar{x}+1)\bar{y}] = \bar{y}$]

ii) $f = \bar{y} - 1$ $\qquad (\bar{y}-1) = (\bar{x}, \bar{y}-1)^2$

["$\supseteq$": only $\bar{x}^2 \in (\bar{y}-1)$ is non-trivial, and

$\bar{x}^2 = 1 - \bar{y}^2 = -(\bar{y}-1)(\bar{y}+1)$

"$\subseteq$": $-\frac{1}{2}[\underbrace{(\bar{y}-1)^2 + \bar{x}^2}_{\in (\bar{y}-1, \bar{x})^2}] = -\frac{1}{2}[\cancel{\bar{y}^2} - 2\bar{y} + 1 + 1 - \cancel{\bar{y}^2}] = 1 - \bar{y}$.
]

Remark: 1) Actually A is a PID, there is an iso:

$$\begin{cases} \mathbb{C}[t]_{(t)} \xrightarrow{\sim} \mathbb{C}[x,y]/(x^2+y^2-1) \\ t \longmapsto x+iy \\ t^{-1} \longmapsto x-iy \end{cases}$$
, and $\mathbb{C}[t]_{(t)}$ is a PID as localization of a PID.

... Dedekind domain with class

2) $A' := \mathbb{R}[x,y]/(x^2+y^2-1)$ is also a Dedekind domain, with class group $\mathbb{Z}/2\mathbb{Z}$ [Non-trivial to prove, skipped]